Einstein-Aether Theory With and Without Einstein

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The exact static spherically symmetric solutions for pure-aether theory and Einstein-aether theory are presented. It is shown that both theories can deliver the Schwarzschild metric, but only the Einstein-aether theory contains solutions with "almost-Schwarzschild" metrics that satisfy Einstein's experiments. Two specific solutions are of special interest: one in pure-aether theory that derives the attractive nature of gravitation as a result of Minskowski signature of the metric, and one - the Jacobson solution- of Einstein-aether theory with "almost-Schwarzschild" metric and non-zero Ricci tensor.

Introduction

Einstein-aether theory was proposed by Jacobson and Mattingly almost 10 years ago [1] (see latest review [2]) as a correction (extension) of Einstein's GR. It postulates that gravitation, in addition to a curved space, is characterized by a unit vector G_i ($G_iG_jg^{ij} = 1$), which together with metric tensor g_{ij} constitute the dynamic variables of the Einstein-aether theory. The equations of motion for g_{ij} and for G_i are derived by variational method from a Lagrangian that is a function of the metric tensor g_{ij} and the vector field G_i.

With the requirement that the equations of motion are of the second order, the most general form of Lagrangian can be written as [1]:

$$S = \int L(g_{ij}, G_i)\sqrt{-g}d^4x$$
$$L = c_0R + c_1G_{i;j}G^{i;j} + c_3G_{i;j}G^{j;i} + c_2(G^i_{;i})^2 + c_4G^{k;i}G_iG_{k;j}G^j$$
$$+T(G_iG_jg^{ij} - 1) \qquad (1)$$

where all c's are constants, T is the Lagrange multiplier, and R is Ricci scalar.

The main question that Jacobson and Mattingly raised was: what should the value of the constants "c"s be in order for the Einstein-aether

theory to yield the same results as predicted by GR? In particular, in case of spherical symmetry the Einstein-aether theory must yield - in the first order of approximation - the Schwarzschild's metric.

The Lagrangian of the aether theory, eq. (1), actually represens three separate theories:

a) General Relativity $c_0 \neq 0$ all other c's are zero ($c_1 = c_2 = c_3 = c_4 = 0$)

b) Einstein-aether theory $c_0 \neq 0$ and at least one of others c's (typically c_1) is not zero

c) Pure-aether theory with $c_0 = 0$ and at least one of others c's is not zero

One question that has not been addressed previously is the validity of pure-aether theory. In other words, do we need Einstein's term ($c_0 R$) in the Lagrangian? One might argue that the Einstein's term ($c_0 R$) is not needed and the pure-aether theory is sufficient to explain the experimental results - at least as far as the solar system is concerned.

In this paper we would like to investigate this issue based on spherically symmetric solutions and we will show that on one hand the Einstein term is not needed - if one wants to get Schwarzschild's metric solution - and on the other hand, the Einstein term leads to unique solutions that are not present in pure-aether theory.

Gravitation. Static, spherically symmetric solution.

In a case of spherical symmetry the last term of Lagrangian (the c_4 term) can be expressed thru the other terms (see [3]) and thus can be dropped out.

The remaining four terms can be written in the following manner:

$$\begin{aligned} S &= \int L(g_{ij}, G_i)\sqrt{-g}d^4x \\ L &= -[\lambda_0 R + \lambda_1 g^{ik} g^{jl} (\partial_j G_i - \partial_i G_j)(\partial_l G_k - \partial_k G_l) \\ &\quad + \lambda_2 R_{ij} G^i G^j + \lambda_3 (G^k_{;k})^2 + T(G_i G_j g^{ij} - 1)] \end{aligned} \tag{2}$$

where the λs are new constants that are linear combinations of c-constants.

The equations of motion for the metric g_{ij} and for the vector field G_i have been obtained previously [3]. However, it is quite a laborious task to derive the exact solution using these equations. Fortunately, in case of spherical symmetry, there is another approach that yields the desired result with almost no hard labor involved.

In a case of static (time independent) spherically symmetric solution the aether theory is governed by 5 functions: g_{00} (or g_{tt}), g_{11} (or g_{rr}), G_0 (or Q_t), G_1 (or Q_r) and T, which are all functions of radius only. We assume that $g_{22} = -r^2$

By using the constraint $G_i G^i = 1$, one can eliminate two of these five functions - g_{00} and T. Our next step is to write the Lagrangian thru the three remaining functions (G_1, g_{11} and G_0). And then find the equations for these functions by means of variation of the Lagrangian with respect to these functions.

The problem is simplified if we choose new variables in this manner:

$$\begin{aligned} &\hat{g} = -g_{11}g_{00} \quad \textit{instead of } g_{11} \\ &\bar{G}_1 = G_1\frac{\sqrt{g_{00}}}{\sqrt{-g_{11}}} \quad \textit{instead of } G_1 \\ &x = \frac{1}{r}; \quad \frac{d}{dx} = ()' \end{aligned} \tag{3}$$

In these new variables g_{00} can be written as :

$$g_{00} = (G_0)^2 - (\bar{G}_1)^2 \tag{4}$$

and the Lagrangian (eq. 2) has this (amazingly) simple form (for details see appendix A):

$$\begin{aligned} &S = -\int d^4v\sqrt{-g}L = \int dtd\Omega S_r, \quad where \\ &S_r = -\int r^2 dr\sqrt{\hat{g}}L(g_{ij}, G_i) = \int \frac{dx}{x^4}\sqrt{\hat{g}}L = \\ &\int dx\{\lambda_0[\frac{2}{\sqrt{\hat{g}}}(\frac{G_0^2}{x})' - \frac{2\sqrt{\hat{g}}}{x^2}] - \frac{2\lambda_1(G_0')^2}{\sqrt{\hat{g}}} - \frac{2\bar{\lambda}_2}{\sqrt{\hat{g}}}(\frac{\bar{G}_1^2}{x})' + \frac{\lambda_3}{\sqrt{\hat{g}}}[x^2(\frac{\bar{G}_1}{x^2})']^2\} \end{aligned} \tag{5}$$

where $\bar{\lambda}_2 = \lambda_2 + \lambda_0$

We now consider the solution of this problem with the following boundary conditions:

a) G_i at infinity has only the time component:

$$G_{i|x=0\ or\ r=\infty} = (1,0,0,0) \quad or$$
$$G_{0|x=0} = 1; \quad G_{1|x=0} = \bar{G}_{1|x=0} = 0 \tag{6}$$

b) metric at infinity corresponds to a flat space:

$$g_{ij\,|x=0\ or\ r=\infty} = diag(1,-1,-r^2,-r^2 sin^2\theta) \tag{7}$$

The requirement that metric g_{ij} satisfies two Einstein experiments (bending of light and precession of Mercury) set these conditions on $\hat{g}$ (bending of light) and g_{00} (precession of Mercury) as functions of x (1/r) with $x \approx 0$ (r$\rightarrow \infty$) [4]:

$$\hat{g} = 1 + \bar{c}x^2 + ...(\bar{c} = constant)$$
$$g_{00} = 1 + c_1 x + c_2 x^3 + ...(c_1\, and\, c_2 = constant) \tag{8}$$

In other words, g_{00} has no quadratic terms in seria by x and $\hat{g}$ has no linear terms.

<u>The case of General Relativity</u>

Let us - mostly to demonstrate the simplicity of this approach and as a sanity check - first consider the case of GR ($\lambda_1 = \lambda_2 = \lambda_3 = 0$).

Variation of eq. (5) with respect to G_0 and $\hat{g}$ yields:

$$\frac{\delta S_r}{\delta G_0} = 0 \quad -> \quad \hat{g}' = 0 \quad or \quad \hat{g} = 1$$
$$\frac{\delta S_r}{\delta \hat{g}} = 0 \quad -> \quad (\frac{G_0^2}{x})' + \frac{1}{x^2} = 0 \quad or \quad g_{00} = G_0^2 = 1 + xC_0 \tag{9}$$

The above solutions are exactly the expression of the Schwarzschild metric. It is worth pointing out that the sign of the constant C_0 in

GR is not set by the theory, but is taken as a seperate postulate that gravitation is only attractive. The GR theory by itself does allow both attraction ($C_0 < 0$) and repulsion ($C_0 > 0$).

The case of Pure-aether theory

Let us now consider the case of pure aether theory ($\lambda_0 = 0$). Variation of eq. (5) with respect to G_0 $\bar{G}_1$, and $\hat{g}$ yields the following equations:

$$\begin{aligned} &a) \quad \frac{\delta S_r}{\delta G_0} = 0 \quad \rightarrow \quad (\frac{G_0'}{\sqrt{\hat{g}}})' = 0 \\ &b) \quad \frac{\delta S_r}{\delta \bar{G}_1} = 0 \quad \rightarrow \quad 2\lambda_2(\frac{1}{\sqrt{\hat{g}}})'\frac{\bar{G}_1}{x} - \frac{\lambda_3}{x^2}[\frac{x^4}{\sqrt{\hat{g}}}(\frac{\bar{G}_1}{x^2})']' = 0 \\ &c) \quad \frac{\delta S_r}{\delta \hat{g}} = 0 \quad \rightarrow \quad \lambda_1(G_0')^2 + \lambda_2(\frac{\bar{G}_1^2}{x})' - \frac{\lambda_3}{2}[x^2(\frac{\bar{G}_1}{x^2})']^2 = 0 \end{aligned} \tag{10}$$

The first equation (eq. 10a) can be integrated and with the third equation (eq. 10c) it can be used to find the function $\hat{g}$.

$$\begin{aligned} &a) \quad G_0' = C_0\sqrt{\hat{g}} \quad C_0 = constant \\ &b) \quad \lambda_3[\bar{G}_1'' - \frac{2\bar{G}_1}{x^2}] = [\frac{\lambda_3}{2}(\bar{G}_1' - \frac{2\bar{G}_1}{x}) - \lambda_2\frac{\bar{G}_1}{x}]\frac{\hat{g}'}{\hat{g}} \\ &c) \quad \hat{g} = -\frac{\lambda_2}{C_0^2\lambda_1}(\frac{\bar{G}_1^2}{x})' + \frac{\lambda_3}{2C_0^2\lambda_1}[x^2(\frac{\bar{G}_1}{x^2})']^2 \end{aligned} \tag{11}$$

It is not difficult to see that if λ_3 is not zero the system of eqs. (11) has no solutions that satisfy the conditions (8). Indeed, if $\hat{g} \approx 1 + x^2\bar{c}$ then $\hat{g}' \approx x\bar{c}$ and the rhs of eq. (11b) is about constant or zero. In order for the lhs of (11b) to be regular, $\bar{G}_1$ is about x^2. In this case the $\hat{g}$ per eq. (11c) should be zero at x=0, which contradicts to the requirements (8).

With $\lambda_3 = 0$, the system of equations (11) can be easily integrated

to give this result:

$$\hat{g} = 1; \quad \bar{G}_0 = 1 + C_0 x; \quad \bar{G}_1^2 = C_1 x - \frac{\lambda_1}{\lambda_2} C_0^2 x^2$$
$$and \quad g_{00} = 1 + (2C_0 - C_1)x + (1 + \frac{\lambda_1}{\lambda_2}) C_0^2 x^2 \tag{12}$$

In the above expressions C_0 and C_1 are constant and $C_1 > 0$.

In order to satisfy the requirements of Einstein's experiments, eq. (8), the quadratic term must be set to zero, which can be achieved if $C_0 = 0$ or $\lambda_1 + \lambda_2 = 0$. In both cases the metric is the Schwarzschild one.

If one sets $C_0 = 0$ the time component of vector field is one ($Q_0 = 1$) and the radius component $\bar{G}_1$ is inverse to square root of the radius r :

$$G_0 = 1; \; g_{00} = -\frac{1}{g_{11}} = 1 - \frac{C_1}{r}; \; G_1 = \sqrt{\frac{\bar{C}_1}{r(1 - C_1/r)^2}} \tag{13}$$

In addition to Schwarzschild metric the pure-aether theory (with the condition $\lambda_3 = 0$) delivers the requirement that gravitation must be attractive, which is the consequence of Minkowski signature of the metric tensor.

Einstein-aether theory ($\lambda_0 \neq 0$)

We now can consider the Einstein-aether theory, or the case when λ_0 is not zero. The presence of Einstein term ($\lambda_0 R$) in Lagrangian significantly changes the number of possible solutions and the choice of λ parameters.

The variation of action integral S_r, eq. (5), leads to this set of equations:

$$
\begin{aligned}
&a) \quad \frac{\delta S_r}{\delta G_0} = 0 \quad \rightarrow \quad \lambda_1 (\frac{G_0'}{\sqrt{\hat{g}}})' - \lambda_0 \frac{G_0}{x} (\frac{1}{\sqrt{\hat{g}}})' = 0 \\
&b) \quad \frac{\delta S_r}{\delta \bar{G}_1} = 0 \quad \rightarrow \quad 2\bar{\lambda}_2 (\frac{1}{\sqrt{\hat{g}}})' \frac{\bar{G}_1}{x} - \frac{\lambda_3}{x^2} [\frac{x^4}{\sqrt{\hat{g}}} (\frac{\bar{G}_1}{x^2})']' = 0 \\
&c) \quad \frac{\delta S_r}{\delta \hat{g}} = 0 \quad \rightarrow \qquad (14) \\
&\frac{\lambda_0}{\sqrt{\hat{g}^3}} (\frac{G_0^2}{x})' + \frac{\lambda_0}{x^2 \sqrt{\hat{g}}} - \frac{\lambda_1}{\sqrt{\hat{g}^3}} (G_0')^2 - \frac{\bar{\lambda}_2}{\sqrt{\hat{g}^3}} (\frac{\bar{G}_1^2}{x})' + \frac{\lambda_3}{2\sqrt{\hat{g}^3}} [x^2 (\frac{\bar{G}_1}{x^2})']^2 = 0
\end{aligned}
$$

From the last equation (14c) one can express $\hat{g}$ as a function of G_0 and $\bar{G}_1$:

$$
\hat{g} = x^2 \{ -(\frac{G_0^2}{x})' + \frac{\lambda_1}{\lambda_0} (G_0')^2 + \frac{\bar{\lambda}_2}{\lambda_0} (\frac{\bar{G}_1^2}{x})' - \frac{\lambda_3}{2\lambda_0} [x^2 (\frac{\bar{G}_1}{x^2})']^2 \} \qquad (15)
$$

Because of the x^2 factor in front of the figure braket it is not difficult to see that for any λs the expression for $\hat{g}$ is always regular (no singularities).

Let us note that if λ_3 is not zero from the equation (14b) follows that for $x \rightarrow 0$ $\bar{G}_1 \approx x^2$ and the λ_2, λ_3 terms of eq. (15) are about x^4.

This means that if we are interested in the behavior of $\hat{g}$ near $x = 0$ the λ_2, λ_3 terms of eq. (15) could be dropped out. The remaining expression for $\hat{g}$ (function of G_0 only) always has a right behavior that satisfies the condition of eq. (8) at x=0. Indeed for $\hat{g}$ we have:

$$
\hat{g} = G_0^2 - 2xG_0G_0' - \lambda_1 x^2 (G_0')^2 \qquad (16)
$$

If we write G_0 near zero as a series by x , $G_0 = 1 + ax + bx^2 + ..$, (a and b are constants) and substitute it in eq. (16) above we will get this approximation for $\hat{g}$:

$$
\begin{aligned}
&\hat{g} = (1 + ax + bx^2 + ...)^2 - 2x((1 + ax + bx^2 + ...)(a + 2bx + ...) \\
&-\lambda_2 x^4 A^2 - \lambda_3 x^4 A^2 = 1 + (-a^2 - 2b - \lambda_1 a^2) x^2 + ... \qquad (17)
\end{aligned}
$$

which as we see has no linear term and thus satisfies the condition (8).

In general the two equations, eq. (14), that describe variables G_0, $\bar{G}_1$ are coupled thru $\hat{g}$, which depends on both functions. There are however four cases where the equations can be uncoupled and there solutions can be presented in analytical forms:

Case A: $\lambda_3 = 0$ and $\bar{G}_1 \neq 0$
Case B: $\bar{G}_1 = 0$ any λ_1, λ_2, λ_3
Case C: $\bar{\lambda}_2 = 0$ (or $\lambda_2 = -\lambda_0$)
Case D: $\lambda_1 = 0$

Case A

If λ_3 is zero (and $G_1 \neq 0$), the equation (14b) yields that $\hat{g} = 1$ and from the equation (14a) follows that G_0 is a linear function of x ($G_0 = 1 + C_0 x$).

The third equation (15) can be used to determine $\bar{G}_1$:

$$\hat{g} = G_0^2 - 2xG_0G_0' + \frac{\lambda_1}{\lambda_0}x^2(G_0')^2 + \frac{\bar{\lambda}_2}{\lambda_0}x^2(\frac{\bar{G}_1^2}{x})' \quad or$$
$$1 = (1 + C_0x)^2 - 2x(1 + C_0x)C_0 + \frac{\lambda_1}{\lambda_0}x^2C_0^2 + \frac{\bar{\lambda}_2}{\lambda_0}x^2(\frac{\bar{G}_1^2}{x})'$$
$$\bar{G}_1^2 = C_1x + C_0^2x^2(\frac{\lambda_1 - \lambda_0}{\bar{\lambda}_2}) \tag{18}$$

And for g_{00} we get:

$$g_{00} = G_0^2 - \bar{G}_1^2 \quad or$$
$$g_{00} = 1 + x(2C_0 - C_1) + C_0^2x^2(1 - \frac{\lambda_1 - \lambda_0}{\bar{\lambda}_2}) \tag{19}$$

This is practically (except for the value of the constants λs) the same result as for pure-aether theory that we derived above - eq.(12).

Case B, ($G_1 = 0$) and C, ($\bar{\lambda}_2 = 0$)

In both of these cases the system of equations (14) can be solved analytically.

In the "case B" ($G_1 = 0$ and thus $\bar{G}_1 = 0$) the eq. (14b) is satisfied and in the remaining two equations λ_2 and λ_3 terms could be dropped, leaving these equations for G_0 and $\hat{g}$:

$$\hat{\lambda}_1(\frac{G_0'}{\sqrt{\hat{g}}})' - \frac{G_0}{x}(\frac{1}{\sqrt{\hat{g}}})' = 0 \quad or \quad G_0'' = (\frac{G_0'}{2} - \frac{G_0}{2\hat{\lambda}_1 x})\frac{\hat{g}'}{\hat{g}} \tag{20}$$

$$\hat{g} = G_0^2 - 2xG_0G_0' + \hat{\lambda}_1 x^2 (G_0')^2 \quad where \quad \hat{\lambda}_1 = \frac{\lambda_1}{\lambda_0} \tag{21}$$

In the "case C" ($\bar{\lambda}_2 = 0$) the eq. (14b) has the solution $\bar{G}_1 = C_1 x^2$ (C_1 - constant), while G_0 and $\hat{g}$ are defined by the same set of equations (20), (21) as in "case B".

The equations (20), (21) - although in slightly different form - had been obtained and investigated by Jacobson in his 2006 paper [1]. The equations can be integrated analytically to yield a result in a form $x = f(G_0)$ (for details see Appendix B):

$$C_0 x = G_0[(G_0^{-\mu} - G_0^{\mu}], \; where \quad \mu = \sqrt{1 - \frac{\lambda_1}{\lambda_0}}$$

$$\hat{g} = \frac{4\mu^2 G_0^2}{[(1-\mu)G_0^{-\mu} - (1+\mu)G_0^{\mu}]^2} \tag{22}$$

where C_0 is a constant equivalent to Schwarzschild radius.

By direct calculation it is not difficult to show that for small x ($x \approx 0$) $\hat{g}$ has no linear terms ($\hat{g} = 1 + ax^2 + ...$) and metric has no quadratic terms ($g_{00} \equiv G_0^2 \approx 1 - C_0 x + bx^3 + ...$) thus satisfying requirements of the Einstein experiments, eq. (8), for any parameter μ.

The behavior of G_0 vs. x outside $x = 0$ (small distance r) significantly depends on a sign of λ_1 (we assume - as in GR - $\lambda_0 > 0$).

If λ_1=0 (the case of GR) the eqs.(22) and (4) yield:

$$C_0 x = 1 - G_0^2 \quad \hat{g} = 1 \quad G_1 = 0$$

$$g_{00} = G_0^2 = 1 - C_0 x \quad C_0 = const \tag{23}$$

with the horizon point at $x = 1/C_0$.

If $\lambda_1 < 0$, x as a function of G_0 monotonically increases as G_0 decreases from 1 to 0. This means that G_0 as a function of x monotonically decreases from 1 to 0 as x changes from x=0 to $x = \infty$.

For the "case B" the metric, defined as $g_{00} = G_0^2$, also decreases from 1 to 0 as x changes from 0 to infinity (the horizon point is $x = \infty$ or r=0). For small G_0 the second term in eq. (22) could be dropped giving this expression for G_0 as a function of x:

$$C_0 x \approx G_0^{1-\sqrt{(1-\hat{\lambda}_1)}}$$
$$G_0 \approx (\frac{1}{C_0 x})^{\frac{1}{\sqrt{(1-\hat{\lambda}_1)}-1}} = (\frac{r}{C_0})^{\frac{1}{\sqrt{(1-\hat{\lambda}_1)}-1}} \quad \hat{\lambda}_1 \equiv \frac{\lambda_1}{\lambda_0} < 0 \tag{24}$$

In the "case C", on the other hand, the metric has additional term: $g_{00} = G_0^2 - (C_1)^2 x^4$, which always - due to Minkowski signature -leads to existence of a horizon point a some point x, the value of which depends on the value of the constant C_1.

If λ_1 positive ($1 > \hat{\lambda}_1 > 0$) - analogues to Maxwell theory, x as a function of G_0 has a bell shape between two points $G_0 = 1$ and $G_0 = 0$ with its maximum at some point in between. This means that $G_0(x)$ exists only from $x = 0$ to a certain point -"dead point". It can be explicitly illustrated for the case of $\hat{\lambda}_1 = 8/9$:

$$C_0 x = G_0[(G_0^{-\frac{1}{3}} - G_0^{\frac{1}{3}}] \quad or \quad G_0 = (\frac{1 + \sqrt{1 - 4C_0 x}}{2})^{\frac{3}{2}} \tag{25}$$

with $x = 1/(4C_0)$ being a "dead point".

For the metric again we have two possibilities:

"Case B": $g_{00} = G_0^2$ and the metric exists up to a "dead point", which is not a horizon point, since g_{00} is not zero.

"Case C": $g_{00} = G_0^2 - (C_1)^2 x^4$ and for sufficiently large C_1 metric (g_{00}) reaches zero - horizon point - at some point before the "dead point". One can choose C_1 in such a way that at "dead point" ($x = 1/4C_0$) the time component of the metric (g_{00}) become zero. That would represent the case when the "dead point" is the horizon.

Case D, $\lambda_1 = 0$

We add this case mostly for the sake of completeness. The condition that $\lambda_1 = 0$ is probably non-physical, due to the fact that equations of

motion for the vector field G_i become of the first - instead of second - order: $aR^k_jG_k + b(G^k_{;k})_{,j} = 0$ with a and b being constants. On the other hand we must remember that the parameter λ_1 in our considerations is a combination of two parameters - see eq. (1) - c_1 and c_4, which would canceled each other only in the case of spherical symmetry.

If $\lambda_1 = 0$, the eq. (14a) yields $\hat{g} = 1$ and the equation for $\bar{G}_1$ can be solved to yield $\bar{G}_1 = C_1x^2$. Knowing $\hat{g}$ and $\bar{G}_1$, we can determine the function G_0 from eq. (15):

$$1 = x^2\{-(\frac{G_0^2}{x})' + \frac{\bar{\lambda}_2}{\lambda_0}3C_1^2x^2\} \quad or \quad G_0^2 = 1 + C_0x + \frac{\bar{\lambda}_2}{\lambda_0}C_1^2x^4$$
$$g_{00} = 1 + C_0x + (\frac{\bar{\lambda}_2}{\lambda_0} - 1)C_1^2x^4 \tag{26}$$

General Case, $\lambda's \neq 0$

In the general case (both G_1 and $\lambda's$ are not zero) the solutions have behavior somewhat in between "case C" and "case D". For the small x (large distance r) G_0 linearly decreases, while $\bar{G}_1$ increases (in absolute value) as x^2. As x moves toward large numbers ($r \to 0$) the G_0 starts deviate from Jacobson's solution while $\bar{G}_1$ deviates from x^2.

The same is true for the time component of metric g_{00}. In addition, if radial component of the vector field $\bar{G}_1$ is present (not zero), the metric has horizon point, which is due to the Minkowski signature of the metric.

Discussion and Conclusion

A we saw above the general solution for aether theory is characterized by two parameters C_0 and C_1. The first one, C_0, typically sets the linear dependence of g_{00} as a function of x with x$\to 0$ (r $\to \infty$) and thus can be identified with a Schwarzschild radius. A much more difficult

question is the meaning of the other parameter C_1, which defines the magnitude of radial dependence of aether vector G_1.

We have shown that if one requires from the aether theory to get Schwarzschild solution for the metric, one can choose both pure-aether theory (no Einstein $\lambda_0 R$ term in Lagrangian) and Einstein-aether theory (with $\lambda_0 R$ term) with $\lambda_3 = 0$ parameter.

It also must be pointed out that the Ricci tensor in both pure-aether and Einstein-aether (with exception for the Jacobson solution) theories is always proportional to the constant C_1 - the radial component (G_1 or G_r) of vector field G_i.

There are two particular solutions of the aether theory that deserve special attention.

The first one is the solution of pure-aether or Einstein-aether theory with $\lambda_3 = 0$ and $C_0 = 0$:

$$G_0 = 1; \quad \bar{G}_1 = \sqrt{C_1 x}\ (C_1 > 0) \quad \hat{g} = 1 \quad and \quad g_{00} = 1 - C_1 x. \quad (27)$$

The presence of "hard" matter does not change the time component of the aether field, but only adds the radial component.

In this solution the atractive nature of gravitation is derived from the aether theory and is due to the Minkowski signature of the space metric.

The second one is the Jacobson's solution given by eq. (22):

$$\begin{aligned} &\bar{G}_1 = 0 \\ &C_0 x = G_0[(G_0^{-\mu} - G_0^{\mu}], \ where \quad \mu = \sqrt{(1 - \frac{\lambda_1}{\lambda_0})} > 1 \\ &\hat{g} = \frac{4\mu^2 G_0^2}{[(1-\mu)G_0^{-\mu} - (1+\mu)G_0^{\mu}]^2} \quad and \quad g_{00} = G_0^2 \end{aligned} \quad (28)$$

As r changes from infinity toward zero, G_0 declines from 1 to zero. Here we have that the hard matter "replaces" the aether. This is opposite to the situation in Maxwell electrodynamics where the vector potential increases toward the center of the charge.

The Jacobson's metric, eq. (28), has no horizon (or to be more precise its horizon point is $r = 0$) and it has singularity ($g_{00} = 0$) at $r = 0$, which of course is an artifact of point-mass consideration.

It is also worthwhile to mention that the distance from the point-mass (r=0) to any point along radius is finite.

$$R = \int_0^r \sqrt{-g_{11}}dr = -\int_0^{G_0(r)} \frac{1}{x^2}\sqrt{\frac{\hat{g}}{g_{00}}}\frac{dx}{dG_0}dG_0$$
$$= \int_0^{G_0} \frac{2\mu G_0^{2(\mu-1)}}{(1-G_0^{2\mu})^2}dG_0 < \infty \quad if \quad \mu > 1 \tag{29}$$

It is often required to express metric tensor in conformly-Euclidean system coordinates defined as $ds^2 = \bar{g}_{00}(y)dt^2 - g_c(d\rho^2 + \rho^2 d\Omega)$. For the Jacobson solution this can be done using these formula (Appendix C):

$$G_0 = \left(\frac{1-\frac{\mu}{4}y}{1+\frac{\mu}{4}y}\right)^{\frac{1}{\mu}} \quad where\ y = 1/\rho$$
$$x = \frac{y}{(1-\frac{\mu}{4}y)^{1-\frac{1}{\mu}}(1+\frac{\mu}{4}y)^{1+\frac{1}{\mu}}} \tag{30}$$
$$\bar{g}_{00}(y) \equiv g_{00}(x(y)) = G_0^2 = \left(\frac{1-\frac{\mu}{4}y}{1+\frac{\mu}{4}y}\right)^{\frac{2}{\mu}}$$

As we mentioned above, the Jacobson metric has no singularities. However, when presented in the conformly-Euclidean system coordinates it does have singularity at $y = \mu/4$. The reason for that is clearly seen from the formula x vs. y in eq (30, line2). The x(y) transfers $x = \infty$ (r=0) to $y = \mu/4$ ($\rho = 4/\mu$). The singularity of conformly-Euclidean system coordinates is due to our "bad choice" of system coordinates. Perhaps, the system coordinates with unity coefficient in front of dr^2 ($ds^2 = \bar{g}_{00}(y)dt^2 - d\rho^2 - g_\Omega(\rho)d\Omega^2$) is a better choice. As we showed earlier in eq. (29), the function $g_\Omega \equiv r^2(\rho)$ is regular for all ρ and the parameter ρ is the true distance betwee two points along the radius.

One puzzling issue of Einstein's GR, that still has not been resolved, is the definition of the energy-momentum tensor of gravitation. It seems logical to identify the tensor $E_{ij} \equiv -(R_{ij} + 1/2Rg_{ij})$ (where R_{ij} is the Ricci tensor) as an energy-momentum tensor of the curved space. The Einstein equations

$$R_{ij} - 1/2Rg_{ij} = T_{ij} \quad or \quad Eij + Tij = 0 \tag{31}$$

then can be read in this manner: the total energy-momentum tensor of the system (matter and space) is zero.

The difficulty here comes from consideration of the vacuum: $T_{ij} = 0$ (no matter) and thus $E_{ij} = 0$, which leads to the unconventional (to say the least) statement that in vacuum gravitation has no energy. As we saw in this paper, all the solutions of the pure-aether theory yield Schwarzschild metric, which in its turns sets to zero Ricci (and thus E_{ij}) tensor. In the Einstein-aether theory, on the other hand, this problem is resolved. Most of the solutions - and Jacobson's metric (with $G_1 = 0$) in particular - yield "almost-Schwarzschild" (up to x^2 terms) metric for which R_{ij} (and thus E_{ij}) is not zero.

This seems to be a key factor in resolving the competition between the pure-aether ($\lambda_0 = 0$) and the Einstein-aether ($\lambda_0 \neq 0$) theories in favor of Einstein-aether theory.

One more note on a physical nature of space. In the Einstein-aether theory (as in Einstein's GR) space (metric) is taken as independent physical entity with some energy attached to it. It is expressed in existing "space only" Lagrangian term ($\lambda_0 R$ - the Einstein term). However, in Einstein-aether theory there is another interpretation of space. We can write the Einstein term in this form:

$$L_R = \lambda_0 R G_k G^k \equiv \lambda_0 R \quad due\ to\ G_k G^k = 1 \tag{32}$$

In this form the Einstein term ($\lambda_0 R$) does not represent space as equal to matter entity, but rather a part of the aether (G_i). The metric, that represents the curved space, is now only auxiliary entity that ties together all forms of matter including the aether as gravitational matter.

Appendix A

In this appendix we derive the expression for the Lagrangian of Einstein-Aether theory (see 1) thru variable g_{00}, $\hat{g} := -g_{00}g_{11}$ and G_0 as a function of $x = 1/r$.

In the case of sphirical symmetry the differential of 4-volume dv can be written as:

$$\sqrt{-g}dv = \sqrt{\hat{g}}\, r^2 dr d\Omega dt = -\sqrt{\hat{g}}\, \frac{1}{x^4}\, dx d\Omega dt \tag{33}$$

The action integral can be written as:

$$\begin{aligned} &S = -\int dv L\sqrt{\hat{g}} = \int dt d\Omega S_r \\ &where \quad S_r = -\int r^2 dr L\sqrt{\hat{g}} = \int dx L(x)\sqrt{\hat{g}(x)} \quad x = 1/r \end{aligned} \tag{34}$$

Since Lagrangian is only a function of radius r (or x=1/r), to shorten the formula everywhere below in writing action integral S we will drop the term $d\Omega dt$.

For components of tensor Ricci we have:

$$\begin{aligned} R_{00} &= R^i{}_{0i0} = \Gamma^i_{00,i} - \Gamma^i_{0i,0} + \Gamma^i_{im}\Gamma^m_{00} - \Gamma^i_{0m}\Gamma^m_{0i} \\ &= \Gamma^1_{00,1} + [\Gamma^0_{01} + \Gamma^1_{11} + 2\Gamma^2_{21}]\Gamma^1_{00} - [\Gamma^1_{00}\Gamma^0_{01} + \Gamma^0_{01}\Gamma^1_{00}] \\ &= \Gamma^1_{00,1} + [-\frac{g_{00,1}}{2g_{00}} + \frac{g_{11,1}}{2g_{11}} + \frac{g_{22,1}}{g_{22}}]\Gamma^1_{00} \\ &= (\frac{\Gamma^1_{00}\sqrt{-g_{11}}g_{22}}{\sqrt{g_{00}}})_{,1}\frac{\sqrt{g_{00}}}{\sqrt{-g_{11}}g_{22}} \\ &= \frac{g_{00}}{2\hat{g}r^2}(\frac{g_{00,1}r^2}{\sqrt{\hat{g}}})_{,1} = \frac{x^4 g_{00}}{2\sqrt{\hat{g}}}(\frac{g'_{00}}{\sqrt{\hat{g}}})' \end{aligned} \tag{35}$$

$$\begin{aligned} &R_{11} = R^i{}_{1i1} = \Gamma^i_{11,i} - \Gamma^i_{1i,1} + \Gamma^i_{im}\Gamma^m_{11} - \Gamma^i_{1m}\Gamma^m_{1i} \\ &= \Gamma^1_{11,1} - [\Gamma^0_{10,1} + \Gamma^1_{11,1} + 2\Gamma^2_{12,1}] \\ &+[\Gamma^0_{10} + \Gamma^1_{11} + 2\Gamma^2_{12}]\Gamma^1_{11} - [\Gamma^0_{10}\Gamma^0_{10} + \Gamma^1_{11}\Gamma^1_{11} + 2\Gamma^2_{12}\Gamma^2_{12}] \\ &= -\Gamma^0_{10,1} - [\Gamma^0_{10} - \Gamma^1_{11} + 2\Gamma^2_{12}]\Gamma^0_{01} + \underline{\underline{2\Gamma^2_{21}(\Gamma^0_{01} + \Gamma^1_{11})}} - 2\underline{[\Gamma^2_{12,1} + \Gamma^2_{12}\Gamma^2_{12}]} \end{aligned}$$

Double underlined terms can be expressed through new variable $\hat{g}$ and the single underlined terms cancel each other.

$$
\begin{aligned}
&= -\Gamma^0_{10,1} - [\Gamma^0_{10} - \Gamma^1_{11} + 2\Gamma^2_{12}]\Gamma^0_{01} + 2\Gamma^2_{21}(\frac{\hat{g}_{,1}}{2\hat{g}}) \\
&= -(\frac{\Gamma^0_{01}\sqrt{g_{00}}g_{22}}{\sqrt{-g_{11}}})_{,1}\frac{\sqrt{-g_{11}}}{\sqrt{g_{00}}g_{22}} + 2\Gamma^2_{21}(\frac{\hat{g}_{,1}}{2\hat{g}}) \\
&= \frac{g_{11}}{2\hat{g}r^2}(\frac{g_{00,1}r^2}{\sqrt{\hat{g}}})_{,1} + (\frac{\hat{g}_{,1}}{r\hat{g}}) = \frac{x^4 g_{11}}{\sqrt{\hat{g}}}[(\frac{g'_{00}}{2\sqrt{\hat{g}}})' + \frac{g_{00}\hat{g}'}{x\hat{g}\sqrt{\hat{g}}}] \\
&= \frac{x^4 g_{11}}{\sqrt{\hat{g}}}[(\frac{g'_{00}}{2\sqrt{\hat{g}}} - \frac{2g_{00}}{x\sqrt{\hat{g}}})' + (\frac{g_{00}}{x})'\frac{2}{\sqrt{\hat{g}}}]
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
R_{22} &= R^i{}_{2i2} = \Gamma^i_{22,i} - \Gamma^i_{2i,2} + \Gamma^i_{im}\Gamma^m_{22} - \Gamma^i_{2m}\Gamma^m_{2i} \\
&= \Gamma^1_{22,1} - \Gamma^3_{23,2} + [\Gamma^0_{01} + \Gamma^1_{11} + 2\Gamma^2_{21}]\Gamma^1_{22} - [2\Gamma^1_{22}\Gamma^2_{21} + \Gamma^3_{32}\Gamma^3_{32}] \\
&= \Gamma^1_{22,1} + [\frac{g_{00,1}}{2g_{00}} + \frac{g_{11,1}}{2g_{11}}]\Gamma^1_{22} + [-\Gamma^3_{23,2} - \Gamma^3_{32}\Gamma^3_{32}] \\
&= \frac{(\Gamma^1_{22}\sqrt{-g_{11}g_{00}})_{,1}}{\sqrt{-g_{11}g_{00}}} - [(\frac{g_{33,2}}{2g_{33}})_{,2} + (\frac{g_{33,2}}{2g_{33}})^2] \\
&= \frac{1}{\sqrt{\hat{g}}}(\frac{g_{00}r}{\sqrt{\hat{g}}})_{,1} - [\frac{(sin^2\theta)_{,\theta}}{2sin^2\theta}]_{,\theta} - [\frac{(sin^2\theta)_{,\theta}}{2sin^2\theta}]^2 = \frac{1}{\sqrt{\hat{g}}}(\frac{g_{00}r}{\sqrt{\hat{g}}})_{,1} + 1 \\
&= \frac{x^4 g_{22}}{\sqrt{\hat{g}}}[(\frac{g_{00}}{x\sqrt{\hat{g}}})' - \frac{\sqrt{\hat{g}}}{x^2}]
\end{aligned} \tag{37}
$$

Combining expressions (35), (36) and (37) we get this expression for the first term (λ_0 term) of the action integral:

$$
\begin{aligned}
S_{r\lambda_0} &= -\int \sqrt{\hat{g}}\, r^2 dr\, \lambda_0 R \\
&= \int \sqrt{\hat{g}}\, dx (x^4)\, \lambda_0[R_{00}g^{00} + R_{11}g^{11} + 2R_{22}g^{22}] \\
&= \int dx\, \lambda_0[(\frac{g'_{00}}{\sqrt{\hat{g}}})' - (\frac{2g_{00}}{x\sqrt{\hat{g}}})' + 2(\frac{g_{00}}{x})'\frac{1}{\sqrt{\hat{g}}} - 2\frac{\sqrt{\hat{g}}}{x^2}]
\end{aligned} \tag{38}
$$

The first two terms in (38) are full differentials and could be dropped from the expression yielding this:

$$S_{r\lambda_0} = \int dx\,\lambda_0[2(\frac{G_0^2 - \bar{G}_1^2}{x})'\frac{1}{\sqrt{\hat{g}}} - 2\frac{\sqrt{\hat{g}}}{x^2}] \tag{39}$$

where per eq. (4) we replace g_{00} with $G_0^2 - \bar{G}_1^2$.

The λ_1-term can be straight forward written as:

$$S_{r\lambda_1} = \int dx\,\lambda_1[-2\frac{(G_0')^2}{\sqrt{\hat{g}}}] \tag{40}$$

For λ_2-term we get the following expression:

$$\begin{aligned}
S_{r\lambda_2} &= -\int r^2 dr\,\lambda_2 R_{ij}G^i G^j = \int x^4 dx\,\lambda_2 R_{ij}G^i G^j \\
&= \int d^4x\,\lambda_2[R_{00}g^{00}(G_0)^2 g^{00} + R_{11}g^{11}(G_1)^2 g^{11}] \\
&= \int dx\,\lambda_2\{(\frac{g_{00}'}{\sqrt{\hat{g}}})'[(G_0)^2 g^{00} + (G_1)^2 g^{11}] + (\frac{g_{00}\hat{g}'}{x\hat{g}\sqrt{\hat{g}}})(G_1)^2 g_{11}\} \\
&= \int dx\,\lambda_2\{\underline{(\frac{g_{00}'}{\sqrt{\hat{g}}})'} + (\frac{g_{00}\hat{g}'}{x\hat{g}\sqrt{\hat{g}}})(G_1)^2 g^{11}\}
\end{aligned}$$

In the expression above the underlined term can be integrated out of this expression. In the second term we switch to the variable $\bar{G}_1 = G_1\frac{\sqrt{g_{00}}}{\sqrt{-g_{11}}}$ and do a partial integration:

$$S_{r\lambda_2} = \int dx\,\lambda_2(\frac{\hat{g}'}{x\hat{g}\sqrt{\hat{g}}})(\bar{G}_1)^2 = \int dx\,(\frac{-2\lambda_2}{\sqrt{\hat{g}}})[\frac{(\bar{G}_1)^2}{x}]' \tag{41}$$

The λ_3 term can be has this form:

$$\begin{aligned}
S_{r\lambda_3} &= -\int r^2\sqrt{g}dr\,\lambda_3(G^k_{;k})^2 = -\int r^2\sqrt{g}dr\,\lambda_3[\frac{(G_{11}g^{11}\sqrt{g})_{,1}}{\sqrt{g}}]^2 \\
&= \int dx\,\frac{\lambda_3}{\sqrt{\hat{g}}}[(\frac{\bar{G}_1}{x^2})'x^2]^2
\end{aligned} \tag{42}$$

Combining the expressions (39) for $S_{r\lambda_0}$, (40) for $S_{r\lambda_1}$, (41) for $S_{r\lambda_2}$ and (42) for $S_{r\lambda_3}$, and introducing $\bar{\lambda}_2 = \lambda_2 + \lambda_0$ we get this final expression for the action integral S_r:

$$S_r = \tag{43}$$
$$\int dx\{\lambda_0[\frac{2}{\sqrt{\hat{g}}}(\frac{G_0^2}{x})' - \frac{2\sqrt{\hat{g}}}{x^2}] - \frac{2\lambda_1(G_0')^2}{\sqrt{\hat{g}}} - \frac{2\bar{\lambda}_2}{\sqrt{\hat{g}}}(\frac{\bar{G}_1^2}{x})' + \frac{\lambda_3}{\sqrt{\hat{g}}}[x^2(\frac{\bar{G}_1}{x^2})']^2\}$$

Appendix B

In this appendix we derive the result of Einstein-Aether theory for the case when $\bar{G}_1 = 0$.

Variation of Lagrangian eq.(4) yields this set of equations:

a) with respect to $\bar{G}_1$ ($\delta S_r/\delta\bar{G}_1 = 0$)

$$2\bar{\lambda}_2\frac{\bar{G}_1}{x}(\frac{1}{\sqrt{\hat{g}}})' + \lambda_3[\frac{x^4}{\sqrt{\hat{g}}}(\frac{\bar{G}_1}{x^2})']'\frac{1}{x^2} = 0 \tag{44}$$

which is satisfied if $\bar{G}_1 = 0$

b) with respect to G_0 ($\delta S_r/\delta G_0 = 0$)

$$\lambda_1(\frac{G_0'}{\sqrt{\hat{g}}})' - \lambda_0\frac{G_0}{x}(\frac{1}{\sqrt{\hat{g}}})' = 0 \quad or$$

$$G_0'' = (\frac{G_0'}{2} - \frac{G_0}{2\hat{\lambda}_1 x})\frac{\hat{g}'}{\hat{g}}; \; where \; \hat{\lambda}_1 = \frac{\lambda_1}{\lambda_0} \tag{45}$$

c) with respect to $\hat{g}$ ($\delta S_r/\delta\hat{g} = 0$)

$$\hat{g} = x^2\{-(\frac{G_0^2}{x})' + \frac{\lambda_1}{\lambda_0}(G_0')^2 + \frac{\lambda_2}{\lambda_0}(\frac{G_1^2}{x})' - \frac{\lambda_3}{2\lambda_0}[x^2(\frac{\bar{G}_1}{x^2})']^2\}$$

$$or\ with\ \bar{G}_1 = 0 \quad \rightarrow \; \hat{g} = G_0^2 - 2G_0G_0'x + \hat{\lambda}_1(G_0')^2x^2 \tag{46}$$

We now introduce a new variable $x = ln(y)$ and write equations (45), (46) as:

$$y = ln(x) \quad G_0' = \dot{G}_0\frac{1}{x} \quad G_0'' = (G_0)^{\cdot\cdot}\frac{1}{x^2} - \dot{G}_0\frac{1}{x^2}$$

$$a) \quad \hat{g} = G_0^2 - 2G_0\dot{G}_0 + \hat{\lambda}_1(\dot{G}_0)^2$$

$$b) \quad (G_0)^{\cdot\cdot} - \dot{G}_0 = (\frac{\dot{G}_0}{2} - \frac{G_0}{2\hat{\lambda}_1})\frac{\dot{\hat{g}}}{\hat{g}} \tag{47}$$

Substituting eq. (47a) in eq. (47b) we will get:

$$[(G_0)^{\cdot\cdot} - \dot{G}_0][G_0^2 - 2G_0\dot{G}_0 + \lambda_1(\dot{G}_0)^2] =$$

$$(\frac{\dot{G}_0}{2} - \frac{G_0}{2\hat{\lambda}_1})[2G_0\dot{G}_0 - 2(\dot{G}_0)^2 - 2G_0(G_0)^{\cdot\cdot} + 2\lambda_1\dot{G}_0(G_0)^{\cdot\cdot}] \tag{48}$$

And after some algeraic manipulations we will get:

$$(G_0)^{\cdot\cdot} G_0^2 = \hat{\lambda}\dot{G}_0^3 - G_0\dot{G}_0^2 + G_0^2\dot{G}_0 \tag{49}$$

The equation above has no explicit y-variable and thus can be reduced to the equation of first order by switching to the new varable $V(G_0) = \dot{G}_0(y)$:

$$\frac{dV}{dG_0} = \hat{\lambda}_1(\frac{V}{G_0})^2 - \frac{V}{G_0} + 1 \tag{50}$$

And after introducing new variable $\bar{V} = V/G_0$:

$$\frac{d\bar{V}}{dG_0}G_0 = \hat{\lambda}_1\bar{V}^2 - 2\bar{V} + 1 \tag{51}$$

which can be integrated:

$$\begin{gathered}\int \frac{d\bar{V}}{\hat{\lambda}_1\bar{V}^2 - 2\bar{V} + 1} = ln(G_0) + C \quad or \\ ln(\frac{\bar{V} - \bar{V}_1)}{(\bar{V} - \bar{V}_2)} = \mu\, ln(G_0) + C \rightarrow \bar{V} = \frac{\bar{V}_1 - C\bar{V}_2 G_0^{\mu}}{1 - CG_0^{\mu}} \\ , \quad where \quad \mu = \hat{\lambda}_1(\bar{V}_1 - \bar{V}_2); \quad \bar{V}_{1,2} = \frac{1\underline{+}\sqrt{1-\hat{\lambda}}}{2\hat{\lambda}}\end{gathered} \tag{52}$$

In the formula (52) above C is an integration constant and $\bar{V}_1$ and $\bar{V}_2$ are the roots of quadratic polymon on rhs of eq.(51). Taking into account the expression for $\bar{V}$ thru $G_0(x)$ we get this equation:

$$\int \frac{dG_0(1 - CG_0^{\mu})}{G_0(\bar{V}_1 - C\bar{V}_2 G_0^{\mu})} = \int \frac{dx}{x} \tag{53}$$

The constant C above must be chosen as $C = \bar{V}_1/\bar{V}_2$ for the reason that lhs of equation above has logafifmic behavior at G_0 near 1 as rhs at x=0.

$$\int \frac{dG_0(1 - \frac{\bar{V}_1}{\bar{V}_2}G_0^{\mu})}{G_0\bar{V}_1(1 - G_0^{\mu})} = \int \frac{dx}{x} \tag{54}$$

Substituting $U = G^{\mu}$ the eq. can be written as:

$$\int \frac{dU(1 - \frac{\bar{V}_1}{\bar{V}_2}U)}{U(1-U)} = \mu \bar{V}_1 ln(x) + C_0$$

$$or \quad \int dU[\frac{1}{U} + \frac{(\frac{\bar{V}_1}{\bar{V}_2} - 1)}{U-1} = ln(C_0 X^{V_1 \mu})$$

$$or \quad U(U-1)^{\frac{\bar{V}_1}{\bar{V}_2}-1} = C_0 x^{\mu V_1}$$

$$or \quad G^{\frac{1}{\bar{V}_1}}(G^{\frac{\bar{V}_1 - \bar{V}_2}{\bar{V}_1 \bar{V}_2}} - 1) = C_0 x \tag{55}$$

And substituting in above the values for $\bar{V}_1$ and $\bar{V}_2$ thru λ we get:

$$G_0(G^{\sqrt{1-\hat{\lambda}}} - G^{-\sqrt{1-\hat{\lambda}}}) = C_0 x \tag{56}$$

The sign of C_0 should be chosen to satisfy the condition of "attractive gravity" - G_0 decreases as x increases from zero on.

Appendix C

The goal of this appendix is to derive the expression for the Jacobson metric in conformly-Euclidean coordinates. The Jacobson metric is a spherically symmetrical metric given by this expression - see eq. (11).

$$
\begin{aligned}
&g_{00} \equiv g_{tt} = G_0^2; \quad C_0 x = G(G^{-\mu} - G^{\mu}) \quad x = \frac{1}{r} \\
&g_{11} \equiv g_{rr} = -\frac{\hat{g}}{g_{00}} \quad \hat{g} = \frac{4\mu^2 G_0^2}{[(1-\mu)G^{-\mu} - (1+\mu)G^{\mu}]^2} \\
&C_0 = const. \quad \mu = \sqrt{1-\lambda} \quad \lambda < 0
\end{aligned}
\tag{57}
$$

The transition to conformly-Euclidean coordinates (r$\rightarrow \rho$) is done according to this equation:

$$
\begin{aligned}
&ds^2 = \bar{g}_{00} dt^2 - (\frac{r}{\rho})^2 dl^2 \quad l = Euclidean\ length \\
&where \quad \bar{g}_{00}(\rho) = g_{00}(r(\rho)) \\
&and\ r(\rho)\ satisfies\ \sqrt{-g_{11}}\, \frac{dr}{d\rho} = \frac{r}{\rho}
\end{aligned}
\tag{58}
$$

The eq. 58 can be first written in x and y coordinates (x=1/r; y=$1/\rho$) and then in G_0, y coordinates:

$$
\begin{aligned}
&\sqrt{\frac{\hat{g}}{g_{00}}}\, \frac{dx}{x} = \frac{dy}{y} \\
&\sqrt{\frac{\hat{g}}{g_{00}}}\, \frac{dx}{x dG_0} dG_0 = \frac{dy}{y}
\end{aligned}
\tag{59}
$$

Substituting expressions eq.(57) in to eq.(59) we will get:

$$
\begin{aligned}
&\frac{2\mu}{G_0[G_0^{\mu} - G_0^{-\mu}]} dG_0 = \frac{dy}{y} \\
&2\frac{du}{u^2-1} = \frac{dy}{y} \quad where \quad u = G_0^{\mu} < 1
\end{aligned}
\tag{60}
$$

or after integration:

$$
\begin{aligned}
& ln(\frac{1-u}{1+u}) = ln(y) + C \\
& or \quad y = \frac{4}{\mu}(\frac{1-G_0^\mu}{1+G_0^\mu}) \quad G_0 = (\frac{1-\frac{\mu}{4}y}{1+\frac{\mu}{4}y})^{\frac{1}{\mu}} \\
& and \quad g_{00} = G_0^2 = (\frac{1-\frac{\mu}{4}y}{1+\frac{\mu}{4}y})^{\frac{2}{\mu}}
\end{aligned} \tag{61}
$$

where the constant C is taken as $C = \frac{\mu}{4}$ so g_{00} at small y (large ρ) has approximation $g_{00} = 1 - y$. From here we can find the transformation coordinates $x \to y$:

$$x = \frac{y}{(1-\frac{\mu}{4}y)^{1-\frac{1}{\mu}}(1+\frac{\mu}{4}y)^{1+\frac{1}{\mu}}} \tag{62}$$

IF $\mu = 1$ (the case of GR) the expression (62) becomes:

$$x = y\frac{1}{(1+\frac{1}{4}y)^2} \quad or \quad r = \rho(1+\frac{1}{4\rho})^2 \tag{63}$$

which is a well know expression from the theory of GR [5].

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